$$\begin{aligned}
&1 \text{ elector:} && A \to B \to C \\
&1 \text{ elector:} && A \to C \to B \\
&2 \text{ electors:} && C \to A \to B \\
&1 \text{ elector:} && B \to A \to C
\end{aligned}$$

As can be seen, no candidate satisfies the Majority criterion. Both A and C are the first choice for two electors out of five. With Single vote, they would obtain 40% of the votes, not enough for the Majority criterion. Hence, no winner is immediately available. So, we apply Condorcet's method and, by carrying out all possible comparisons, we get the following table, which we call *Condorcet table*, in which each number indicates the number of times that the alternative corresponding to that row is preferred to the alternative corresponding to that column. The winning values are denoted in boldface:

	A	B	C
A		**4**	**3**
B	1		2
C	2	**3**	

In this example, A outnumbers both B and C, and so A is the Condorcet winner. With Single vote, we are not able to make a choice between A and C, but having more information, that is to say, all rankings among the alternatives, allows us to go deeper into the decision and make the final choice.

The method produces a winner, and, by proceeding recursively, we may obtain the full ranking. If we drop the row and the column corresponding to the winner from the table, we may simply repeat the procedure on the remaining alternatives. We must point out an important fact while we operate in this way: having or not having the comparisons with the dropped alternative does not alter the other comparisons, and so we observe that *the evaluations on the alternatives are not modified if we drop, or possibly add, alternatives*. Similarly, a Condorcet winner remains the winner even if we drop alternatives or change the preferences among the other alternatives. We shall come back to this important property later.

In this case, with only three alternatives, the procedure is very simple. Only C and B remain, and, for them, we already have the direct comparison available. Hence, the collective ranking is $A \Rightarrow C \Rightarrow B$.

4.3 Condorcet Cycles

Of course, things are not so simple as they might appear at first sight. Indeed, there may be no Condorcet winner, in other words, it may happen that each alternative is defeated by at least one other alternative. It seems reasonable that this may occur, but

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the baffling aspect is the following: take any alternative and consider the alternative by which it is defeated (this must exist if there is no Condorcet winner). In turn, this alternative is defeated by another one, which, in turn, is also defeated by another one, and so on. Since the number of alternatives is finite, there must be a case in which, at a certain point, we find an alternative that we have already considered, and we have therefore found a preference cycle, which is indeed called a *Condorcet cycle*.

This paradoxical aspect, which Condorcet himself discovered, shows that, while preference cycles are not admitted individually, as we have already discussed, contrastingly, they may occur collectively, due to the different majorities that occur from comparison to comparison. The paradigmatic case, with three electors and three alternatives, is as follows:

$$A \to B \to C, \qquad B \to C \to A, \qquad C \to A \to B,$$

for which there is the cycle $A \Rightarrow B \Rightarrow C \Rightarrow A$. Indeed, A defeats B two times out of three, B defeats C two times out of three and, finally, C defeats A two times out of three. This is a case of total symmetry among the alternatives, and there can be no method that expresses a winner with the available information.

With a large number of electors, the probability of symmetries among the alternatives is very low. Hence, we may wonder if, barring symmetries, it is possible to use the available information to establish a winner anyway, even in the presence of cycles. Imagine the following example, which has been obtained by slightly modifying the previous example:

$$\begin{aligned} &\text{2 electors:} && A \to B \to C \\ &\text{2 electors:} && C \to A \to B \\ &\text{1 electors: :} && B \to C \to A \end{aligned}$$

This yields the Condorcet table

	A	B	C
A		**4**	2
B	1		**3**
C	**3**	2	

There is no Condorcet winner, because each alternative is defeated by some other alternative and the cycle $A \Rightarrow B \Rightarrow C \Rightarrow A$ exists.

Over time, several methods have been proposed for overcoming the deadlock caused by the presence of a cycle [54, 97, 104]. Lull himself devised one way to solve this problem. He counted the number of victories earned by an alternative in each pairwise comparison (so, in our tables, the number of entries in boldface is counted). The alternative with the most victories is the winner. Unaware of Lull's results, Copeland has proposed the same idea more recently [32], and so this way

of solving the problem is known as Copeland's method. A Condorcet winner is necessarily a Copeland (or Lull) winner. However, this method does not always produce a winner, as our little example, in which all alternatives have one victory each, demonstrates.

The method that seems most valid is quite recent and was proposed by Schulze [85, 86]. The basic idea of the method is based on the observation that, if a cycle is present, beside the direct preference $A \Rightarrow B$ (as in the example), there is another indirect preference consisting of a sequence of preferences $B \Rightarrow C \Rightarrow A$.

How to evaluate these indirect preferences? For instance, if we evaluate the sequence $B \Rightarrow C \Rightarrow A$, we see that three electors prefer B to C and three electors prefer C to A. Thus, we can say that the indirect preference of B over A through the sequence $B \Rightarrow C \Rightarrow A$ is supported by at least three electors, and so the 'strength' of the preference is evaluated as 3. It is useful to note that we do not take into account the common electors shared by the two groups of electors (in this example, there is just one elector in common). The groups may consist of different electors. The only thing that matters is their number. Hence, given a sequence, its strength is computed as the smallest number in the various numbers of the pairwise preferences.

If there are several sequences linking two alternatives, it seems reasonable to evaluate the strength of a preference (either direct or indirect) as the largest value among all possible sequences. This computation, which,, at first sight seems quite complex, can be done very efficiently on a computer. Let us consider the following example, slightly more complex than the previous one (nine electors and four alternatives):

$$\begin{aligned}
&3 \text{ electors:} \quad A \to B \to C \to D \\
&3 \text{ electors:} \quad D \to C \to A \to B \\
&2 \text{ electors:} \quad B \to C \to A \to D \\
&1 \text{ elector:} \quad D \to A \to B \to C
\end{aligned}$$

From these data, we get the following Condorcet table:

	A	B	C	D
A		**7**	4	**5**
B	2		**6**	**5**
C	**5**	3		**5**
D	4	4	4	

which shows the cycle $A \Rightarrow B \Rightarrow C \Rightarrow A$. There are four electors that directly prefer A to C, but the strength of the indirect sequence $A \Rightarrow B \Rightarrow C$ (in this example, there are no other sequences, but, in general, there could be many more) is the minimum between 7 (preference $A \Rightarrow B$) and 6 (preference $B \Rightarrow C$), and so it is 6. This value is higher than the five electors of the direct preference $C \Rightarrow A$, and thus the 'strength' of A versus C is higher than that of C versus A. By completing this type of computation for all pairs of alternatives, we get the following table, which we may call a *Schulze table*, in which the numbers represent the strength of the alternative of

the row with respect to the alternative on the column (for each pair, we denote the winning alternative, i.e., the one with more strength, in boldface)

	A	B	C	D
A		**7**	**6**	**5**
B	5		**6**	**5**
C	5	5		**5**
D	4	4	4	

Thus, the choice falls on A. To complete the ranking, we easily see that, in this example, $A \Rightarrow B \Rightarrow C \Rightarrow D$. It is not difficult to see that a Condorcet winner is also a winner for the Schulze method.

We have to add that the Schulze method might also not produce a winner, because there may be ties, even if, as already stated, there is a very low probability of this with many electors. There are various proposals for a way out from the impasse [85, 86].

4.4 Principle of Independence of Irrelevant Alternatives

So far, it seems that we have solved a deadlock situation in a satisfactory way. However, we have also lost an important property. Let us examine a situation in which, for some reason, an alternative is no longer available. For instance, in a musical contest, a candidate might feel sick and not be able to play. The deletion of an alternative from the Condorcet table can be done easily by simply dropping the row and the column corresponding to that alternative. We don't have to worry about the other numbers in the table, because they have been obtained independently of the presence of the deleted alternative.

If B withdraws in the previous example, a surprising thing happens: C becomes the Condorcet winner. We recall that A was not the Condorcet winner, but turned out to be the winner only after the Schulze method was applied. A Condorcet winner remains the winner independently of the cancellation of alternatives, but, in our case, there was no Condorcet winner with four alternatives.

What about a withdrawal of B after having applied the Schulze method and proclaimed A the winner? Given the ranking $A \Rightarrow B \Rightarrow C \Rightarrow D$ that we have computed, we would very likely continue to consider A the winner, thus violating the Condorcet criterion that we had decided to consider compelling.

How is this paradoxical situation possible? While a Condorcet table does not depend on other choices, a Schulze table does depend on the other comparisons. A prevails over C thanks to the presence of B. It is the seven preferences of A over B and the six preferences of B over C that give us the strength six for A versus C. But, if B disappears, these numbers also disappear and A is defeated against C.

Clearly, this is not a minor issue. Besides, different outcomes may arise from situations other than missing alternatives. If, for instance, the two electors that have expressed the ranking $B \to C \to A \to D$ do not change their opinion between A and C, but do change their opinion between B and C, thus producing the new ranking $C \to B \to A \to D$, the Condorcet table is modified as

	A	B	C	D
A		**7**	4	**5**
B	2		4	**5**
C	**5**	**5**		**5**
D	4	4	4	

from which we see that C is the Condorcet winner and the new collective ranking is $C \Rightarrow A \Rightarrow B \Rightarrow D$ (both for Condorcet and for Schulze). Hence, two electors have changed their preferences for the alternatives B and C, but nobody has changed their preference between A and C, and yet the collective evaluation between A and C has been modified.

The fact that the collective preference between two alternatives may change only because another alternative is or is not present, or some individual preferences of other alternatives are changed, is so important as to deserve formal expression as a new criterion to be observed (if possible):

Criterion of the Independence of irrelevant alternatives: *The collective preference within any subset of alternatives must be invariant with respect to any change of individual preferences outside of the subset.*

Therefore, Condorcet's method, when it produces a winner, does respect the Criterion of the Independence of irrelevant alternatives, while the Schulze method, which yields a winner when Condorcet is not able to do so, does not respect the criterion in general.

4.5 Choice Among Many Motions

Condorcet's method does indeed have the purpose of choosing a motion among many motions that have been proposed in the discussion of an assembly. It is worthwhile once more to insist upon this issue, because it frequently happens in politics that many proposals are presented, but none reach the necessary majority for approval. The recent events in the British parliament that was faced with many proposals concerning the Brexit affair are a clear example of how politics can go into a deadlock with the usual electoral method.

Anyone who is familiar with sport might think of proposing a method that is widely employed in sport competitions, that is, a direct elimination championship,

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